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ON THE RELATIVISTIC THEORY OF ROCKET FLIGHT*

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It is shown by macroscopic analysis that, when the entire mass of a rocket is consumed for creating thrust, objects may be obtained as a result having energy but zero mass, moving the velocity of light. It is shown that the boost process of such massless objects can be realized in finite time from the observer's point of view. The vast stellar luminosity of quasars and certain jet motions observed in remote space can be explained by the production of massless radiation with internal motions connected with the separation of large energies inside the stars.

A number of publications have dealt with rocket motions in the context of relativistic effects. One of the first was by Ackeret /1/, then there was Sanger's /2/, while other authors largely took these as a basis for their first principles. It should be mentioned that some authors have sometimes used super-light relative velocities of the rejected masses, which is not admissible.

Sanger gave a detailed theory of inhabited relativistic rockets with equipages in board, allowing for the arrival at the rocket of opposed cosmic masses, used as energy sources in reactive motors of the stright-through type.

Below we study limit relations for the motions of uninhabited rockets, when their consumed rest mass tends to zero.

A typical feature of rocket motion is connected with the rejection of mass belonging to them and the consequent creation of thrust, so that start and boost of the rocket can be realized with a considerable increase in their flight velocity relative to a fixed observer.

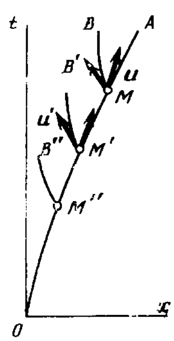

Fig.1

In Fig.1 we show schematically the world line of the rocket in the Cartesian inertial system of observer's reading xt. An arbitrary rocket position is denoted by M; $M''B''$, $M'B'$, and MB are elements of the world lines for the rejected infinitesimal masses $|\Delta m^k|$ by means of the rocket motors, and $\mathbf{u}_k'$ is their initial four-dimensional velocity (the gas flow velocity from the motor nozzles). In general it is natural to assume that the further motion of the rejected masses along world lines MB cannot affect the motion of rocket M, though the properties of the initial vector $\mathbf{u}_k'$ are extremely important.

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Denote by $\mathbf{P} = m_0\mathbf{u}$ the four-dimensional vector of rocket momentum with rest mass m_0 at the given instant t^* of proper time, since the difference $\mathbf{P}(t^* + dt^*) - \mathbf{P}(t^*)$ is largely determined by the removal from the rocket of the momentum of masses $-\Delta m_k$, ejected by the rocket motor with different velocities $\mathbf{u}_k'$ $(k = 1, 2, \ldots)$, in general in different directions.

For the change of initial total rocket momentum $m_0 u = P$ in the context of the momenta removed by the masses Δm_k ejected in time Δt^*, in the local inertial proper reference system at the point M, taken on rocket world line OA, we can write

$$\frac{d\mathbf{P}}{dt^*} = \frac{dm\mathbf{u}}{dt^*} = m\frac{d\mathbf{u}}{dt^*} - \sum_k \frac{dm_k}{dt^*}\mathbf{w}_k = \mathbf{G}, \quad dm = -\sum_k |dm_k| \tag{1}$$

Here, $\mathbf{w}_k$ are the corresponding velocities relative to the rocket, in general of different size and direction of ejected masses dm_k, $\mathbf{G}$ is the total external force applied to the rocket and due to graviation and the forces of rocket interaction with the external medium and the fields in which the rocket moves. The differential dt^* is a scalar, equal to the increment of the rocket proper time.

From Eq. (1), in which it is assumed that $\mathbf{Q} = Q'\mathbf{u} = 0$ at all points of the rocket trajectory OA, where Q' is the inflow of internal non-mechanical energy, it follows that the velocity vectors $\mathbf{w}_k$ have a spatial character, like the rocket acceleration vector $d\mathbf{u}/dt^*$, if the action of the external medium on the rocket is represented by only a vector of the three-dimensional spatial external force $\mathbf{G}$.

The moving elements of mass $|\Delta m_k|$ at the instant t^* rapidly change the direction of their velocity (in the limit, for a particle, by a jump) from the four-dimensional velocity $\mathbf{u}$ to the four-dimensional velocity $\mathbf{u}_k'$. We transfer the terms

$$-\sum_k \frac{dm_k}{dt^*}\mathbf{w}_k$$

from the left- to the right-hand side of Eq. (1). If several motors act on the rocket, we obtain on the right the vector

$$\mathbf{R} = \frac{dm_1}{dt^*}\mathbf{w}_1 + \frac{dm_2}{dt^*}\mathbf{w}_2 + \ldots$$

which can be regarded as the reactive force due in the limit to the shock change $dm < 0$ of the total momentum of mass m when it is ejected (or when the masses adhere to the body, $dm_k > 0$).

By (1), the general local form of the equation for the curvilinear rocket motion in the proper system K^* has the form

$$m_0\frac{d\mathbf{u}}{dt^*} = \sum_k \frac{dm_k}{dt^*}\mathbf{w}_k + \mathbf{G} \tag{2}$$

where m_0 is the running variable rest mass. The transformation of the rest mass m_0 into energy (notably electromagnetic), or the reverse process, can occur by means of various mechanisms. Here we are concerned with the realization of such processes in the rocket motion. In particular, as model examples of physical massless objects for particles possessing energy, we may mention the concepts of neutrinos or photons, which are introduced and considered in physics.

For the small body of a rocket in Newtonian mechanics we use the model of a material particle, to which we allot mass and energy. It is understood below that the total energy of the rocket as a material particle is made up of the energy equal to the kinetic energy $m_0v^2/2$ with respect to the observer, and the internal (thermal, electromagnetic, chemical, or nuclear) energy, which can transform in either direction into or from mechanical energy. Under the action of certain physically admissible internal processes in the modelled body, there can occur according to certain given laws the ejection of mass and energy from the body or the attachment to it of mass and radiation that move in space and are encountered by it.

In this way we can construct a theory of a rocket as particle motion, in which the mass and energy vary according to given laws (as a result of external factors determined by supplementary conditions) while observing all the other mechanical laws both in Newtonian mechanics and in the theory of relativity.

It is important that there may be present in the basic equation of motion (2) for the particle, given vectors of reactive forces, defined by the three-dimensional velocities, relative to the rocket, of ejected or added mass or energy.

We know that, in accordance with the foundations of relativity theory, the velocities $\mathbf{w}_k$ cannot exceed the velocity of light, calculated in the local proper inertial reference systems for the point *M*, so that $|w_k| \leqslant c = 3\cdot 10^{10}$ cm/sec.

The discussion will henceforth be confined to the example of rectilinear rocket motion along the x axis in the global inertial system of the observer tx in Newtonian mechanics and in the special theory of relativity (Fig.1) in the simple example when $\mathbf{G} = 0$ and there is only the one velocity $\mathbf{w}_1 = \mathbf{w}$, with which parts dm_0 of the rocket rest mass are ejected backwards. In this case we can write (2) as

$$m_0\mathbf{a} = (dm_0/dt^*)\,\mathbf{w} = \mathbf{R} \tag{3}$$

where $\mathbf{R}$ is the reactive force. It is clear, in particular, from (3) that the equal vectors $m_0\mathbf{a}$ and $(dm_0/dt^*)\,\mathbf{w}$ have equal components in any basis tetrads.

In any coordinate system, and in particular, in the global inertial coordinate system K_0 of the observer (x, t), Eq.(3) for general rectilinear spatial motion can be written as

$$m_0 \frac{d^2x^i}{dt^{*2}} = \frac{dm_0}{dt^*} w_{x^i} = R_{x^i} \tag{4}$$

Here, m_0, dt^*, and $dm_0/dt^* < 0$ are scalars. Obviously, (4) has the same form both in STR and in Newtonian mechanics, if the proper time t^* is identified with Newtonian absolute time. The subsequent conclusions are just the same as those obtained by Tsiolkovskii's celebrated expression in Newtonian mechanics.

Rectilinear rocket motion along the x axis is governed by the scalar global equation of (4) with index $i = 1$, in which it must also be taken into account that the complete acceleration and velocity $\mathbf{w}$ are directed along the x axis, while $w_x = -w$, where $w = |w|$, since, with $a_x > 0$, the product $(dm_0/dt^*)w_x > 0$, while the projection of the thrust on to the x axis is equal to

$$m_0 \frac{d}{dt^*}\frac{dx}{dt^*} = -\frac{dm_0}{dt^*} w = R \quad (R > 0) \tag{5}$$

The physical magnitude of the velocity $\mathbf{w}(m_0)$ of gas flow relative to the rocket can be specified in various ways, depending on the type and operating mode of the rocket motor, which are linked with internal processes inside the rocket.

We first consider the rocket motion in accordance with Eq.(5) in Newtonian mechanics, when t^* is the absolute time, $dx^*/dt = v$ is the velocity relative to the observer, and $w(m_0)$ is a given function, characterizing the rocket motor operation. We have $m_0 dv = -w dm_0$ and the equation of balance of the masses ejected backwards

$$dm/dt = -kw \tag{6}$$

where k is a parameter characterizing the property of the motor. From (6), when $v = 0, m_0 = m_{00}$ for $t = 0$ and $kw = \text{const}$, we obtain

$$v = -w \ln\frac{m_0}{m_{00}}, \quad t = \frac{m_{00} - m_0}{kw} \tag{7}$$

For the rocket kinetic energy and its total variable energy E, equal to the operation of the reactive thrust on the rocket displacements, we have

$$\frac{m_0 v^2}{2} = \frac{w^2}{2} m_0 \ln^2\frac{m_0}{m_{00}} \tag{8}$$

$$E = \int_0^x R\,dx = \int_0^t Rv\,dt = \frac{m_0 v^2}{2} + \int_{m_0}^{m_{00}} \frac{v^2}{2}\,dm_0 \tag{9}$$

From (7)-(9) we find that, as $m_0 \to 0$, we have the limit relations

$$v \to \mp\infty, \quad \frac{m_0 v^2}{2} \to 0, \quad E \to \frac{w^2}{2}\int_0^{m_{00}} \ln^2\frac{m_0}{m_{00}}\,dm_0 > 0 \tag{10}$$

Thus, in Newtonian mechanics, the velocity of a moving point as a physical object tends to infinity, the kinetic energy tends to zero, and the total energy of the object is finite and non-zero.

The energy of point M is non-zero ($\lim E \neq 0$ as $m_0 \to 0$) because of the inflow of energy from transformation of initial energy partly into energy of ejected masses and from the different kind of energy transformations in the system of rocket motors and partly due to reversion of the initial energy E_0 into energy of the consumed part of the rocket by operation of the thrust force.

Consider the rocket motion when relativistic effects are present in the observer's reference system xt along the x axis. At every point of pseudo-Riemann space, notably, at points of this line, orthonormalized tetrads of the reference system K_0 are defined for the observer with constant bases 3_i. Along with the observer's system, at every point of the x axis we can also introduce a locally proper inertial tetrad as a reference system K_* with orthonormalized bases 3_i^* which are different at every point. For any infinitesimal vector $d\mathbf{r}$ we can write

$$d\mathbf{r} = dl^i 3_i = dl^{*i} 3_i^* \tag{11}$$

We denote the components of $d\mathbf{r}$ in K_0 and K_* in (11), with the respective bases 3_4 and 3_4^*, by cdt and cdt^*, or with bases 3_1 and 3_1^*, by $d\mathbf{l}$ and $d\mathbf{l}^*$, and in accordance with the statement of the problem, we find that the components of $d\mathbf{r}$ in the case of 3_2, 3_2^* and 3_3, 3_3^*, are zero.

Obviously, dt and dt^* are the increments of the observer's time and of the proper time of the moving rocket on a clock with a fixed connection with the inertial tetrads K_0 and K_*.

At every point M of the rocket world line the compounds of $d\mathbf{r}$ in (11) in bases 3_i and 3_i^* are geometrically connected by a transformation which is independent of whether these components are constant or variable. This transformation is the Lorentz, which, however, is defined solely by the three-dimensional variable translational velocity $v = dx/dt$ in the observer's system. In our statement of the problem on the motion of the rocket as a material particle M (Fig.1), the transformation has the form

$$dx = \frac{dx^* + v\,dt^*}{\sqrt{1 - v^2/c^2}}, \quad c\,dt = \frac{c\,dt^* + (v/c)\,dx^*}{\sqrt{1 - v^2/c^2}}$$

$$dx^{*2} = dx^2, \quad dx^{*3} = dx^3$$

We see from these relations that the increments of the rocket clock reading dt^* and the observer's clock reading dt with $dt^* = 0$ are connected by

$$dt^* = dt\sqrt{1 - v^2/c^2}, \; v = dx/dt \tag{12}$$

while the lengths dx^* and dx with $dt^* = dt = 0$ are connected by

$$dx^* = dx\sqrt{1 - v^2/c^2} \tag{13}$$

We can write (5) in the final form, after the replacement $dt^* = dt\sqrt{1 - v^2/c^2}$, as

$$-d\frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \frac{dm_0}{m_0}\,\mathbf{w}(m_0) \tag{14}$$

(It is by means of this replacement that we take account of relativity; as a result of the replacement, the role of infinite rocket velocity in Newtonian mechanics is replaced by the velocity of light in relativistic mechanics.)

Along with Eq.(11) at every rocket position, we can in general introduce vectors with the dimensionality of velocity: $d\mathbf{r}/dt$ and $d\mathbf{r}/dt^*$. In this way, by means of components dl^*, dl, dt^*, and dt, we can introduce four components for two three-dimensional vectors, when the three-dimensional componet of vector $d\mathbf{r}$ is directed along the space axis of the rocket world line

$$w_*^* = \frac{dl^*}{dt^*}, \quad w^* = \frac{dl^*}{dt}, \quad w_* = \frac{dl}{dt^*}, \quad w = \frac{dl}{dt} \tag{15}$$

On the basis of (12)-(15) we have

$$\begin{aligned} &\mathrm{A}\; w_*^* = w, \text{ if } w = c, \text{ then } w_*^* = c \text{ also,} \\ &\mathrm{B}\; w^* = w\sqrt{1 - v^2/c^2} \text{ (with } w^* > c\sqrt{1 - v^2/c^2}\; w_*^* > c) \\ &\mathrm{C}\; w_* = \frac{w}{\sqrt{1 - v^2/c^2}} \end{aligned} \tag{16}$$

Obviously, the components $w_*^* = w$ or $dl^*/dt^* = dl/dt$, which are the rate of out-flow of the working medium in the proper reference system and are precisely equal in magnitude to the rate of out-flow w of the working medium from the observer's point of view, so that, in the limit as $dt^* \to 0$, for the photon rocket $w = c$ and $w_*^* = c$.

Notice that the quantities $w = w_*^*$ are the physical characteristic of the rocket motor operation, its stability and controllability, which must necessarily be specified in calculations of rocket motion.

It is important and useful to note that, in the proper tetrad $\mathbf{w}_*^* = -w_*^* 3_1^*$, while in the observer's system $\mathbf{w} = -w 3_1$. Consequently, in the general case, Eq.A holds only in components.

Passage from $w_*{}^*$ to w occurs not only as a result of a Lorentz transformation, but also as a result of passage from dt^* in the proper reference system K_* to dt in the observer's system K_0.

We mentioned above that it is the velocity $w_*{}^*$ that has a physical meaning in the equation of fuel mass consumption, in fact, it characterizes the working mode of the rocket motor and is worth specifying as an essential parameter in the expression (4) for the thrust, written in the proper tetrad.

From the mathematical point of view, the velocities

$$w_*{}^*,\ w,\ w^*,\ w_* \tag{17}$$

can be specified as the appropriate and distinct functions of m_0 and v, which leads to equivalent problems. The corresponding dependences on the choice of observer and on the motor working modes in fact define the rocket laws of motion. At the same time, the velocities $w_*{}^* = w\,(m_0)$ depend only on the mode of rocket motion. It is worth adding to this that all the velocities in (17) may be variable.

In Ackeret's formula it is understood non-relativistically that $w^* = \text{const}$, and obviously, even the case when $w^* = c$ may be considered. Clearly, depending on the choice of functions (7), (17), which figure in Eqs.(16), a calculation of the rocket motion can be given. However, from the dynamic point of view, calculations in which the inequality $w_*{}^* \leqslant c$ is violated, cannot be regarded as relativistic (see case B).

It should be noted that the law of mass consumption variation in the accompanying system, as in Newtonian mechanics, is governed by the equation

$$dm_0/dt^* = -kw\,(m_0) \tag{18}$$

The solution of Eq.(14) may be started by means of a quadrature; if, with $t = 0$ (at the start of the rocket) we have $v = 0$ and $m_0 = m_{00}$, then

$$-\frac{\mathrm{v}}{\sqrt{1-v^2/c^2}} = \int\limits_{m_{00}}^{m_0} \frac{dm_0}{m_0}\,\mathrm{w}\,(m_0) \tag{19}$$

To evaluate the right-hand integral, we have to specify the functional $\mathrm{w}\,(m_0)$ in the general case, or simply the function $w\,(m_0)$, in the case of rectilinear motion.

Consider two typical examples.

1°. A photon rocket (the mass dm_0 is first transformed into radiation, and ejected with constant velocity c, equal to the velocity of light). In this case, we obtain after integration from (19):

$$-\frac{v/c}{\sqrt{1-v^2/c^2}} = \ln\frac{m_0}{m_{00}}, \quad \text{or} \quad m_0 = m_{00}\exp\left[-\frac{v/c}{\sqrt{1-v^2/c^2}}\right] \tag{20}$$

By (20), with $v = 0$ we have $m_0 = m_{00}$, and with $v = c$ $m_0 = 0$. To evaluate the rocket flight time t^* or t from the value $v = 0$ to $v = c$ we have to use Eq.(18) and Eq.(12), in which the coefficient $k\,(m_0)$ is a rocket motor characteristic connected with its design. With $k = \text{const}$, we obtain as a result extremely simple relations for $w = c$. In this case we have

$$\bar{t}^* = -\frac{1}{kc}\int\limits_{m_{00}}^{0} dm_0 = \frac{m_{00}}{kc} \tag{21}$$

$$\bar{t} = -\frac{1}{kc}\int\limits_{m_{00}}^{0}\frac{dm_0}{\sqrt{1-\lambda^2}} = \frac{m_{00}}{kc}\int\limits_{0}^{1}\exp\left(-\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\frac{d\lambda}{1-\lambda^2}, \quad \lambda = \frac{v}{c} \tag{22}$$

Obviously, $\bar{t}^*$ and $\bar{t}$ are finite. Consequently, after all the rocket mass is consumed, the velocity c of light is reached. The rocket then converts into an object with mass $m_0 = 0$, which will continue its rectilinear motion along the x axis with the speed of light, with $R = 0$ and $dt^* = 0$. The energy of this object is equal to the work of the thrust, which is equal to R in the proper tetrad, while its size in the observer's system on the x axis is equal to $R/\sqrt{1-v^2/c^2}$. During boost, its work is represented by the integral

$$E = \int\limits_0^{\bar{t}}\frac{Rv\,dt}{\sqrt{1-v^2/c^2}} = -m_{00}c^2\int\limits_{m_0}^{0}\frac{\lambda\,dm_0}{\sqrt{1-\lambda^2}} = m_{00}c^2\int\exp\left(\frac{-\lambda}{\sqrt{1-\lambda^2}}\right)\frac{\lambda\,d\lambda}{(1-\lambda^2)^{3/2}} \tag{23}$$

Thus, in the context of classical relativistic mechanics, with the aid of the photon rocket we can obtain an object which is an elementary particle with high energy, proportional to m_{00} and equal to E. It can be shown that, with variable $k(m_0) \neq \text{const}$, the above basic qualitative conclusions retain their force. If we consider a rocket on which, in addition to thrust, there also acts a reactive moment likewise due to mass consumption, then we can obtain an elementary object which is a particle with very high energy and momentum.

2°. We consider the similar motion of a rocket in accordance with Fig.1, for which the velocity in (19) $w(m_0) \leqslant c$, but the equation $w(0) = c$ holds with $m_0 = 0$. With any law of variation $w(m_0)$, instead of (20)-(23) we obtain

$$-\frac{\lambda}{\sqrt{1-\lambda^2}} = \int_{m_{00}}^{m} \frac{dm_0}{m_0} \cdot \frac{w(m_0)}{c} \tag{24}$$

$$t^* = -\int_{m_{00}}^{0} \frac{dm_0}{w(m_0)\,k(m_0)}, \quad \bar{t} = -\int_{m_{00}}^{0} \frac{dm_0}{\sqrt{1-\lambda^2(m_0)}} \cdot \frac{1}{w(m_0)\,k(m_0)}$$

$$E = -m_{00}c^2 \int_{m_{00}}^{0} \frac{\lambda w(m_0)\,dm_0}{\sqrt{1-\lambda^2}}$$

By choosing a function $w(m_0) < c$ we can ensure convergence of the definite integrals in the last three expressions as $\lambda \to 1$, and hence the achievement of the velocity of light by the object obtained from the rocket. The practical possibility of obtaining this kind of law of ejected mass from the rocket seems doubtful. In the theory of motion of continua, however, the possibility of obtaining such effects on their boundaries cannot be discounted.

In short, it has been shown that, by controlling the out-flow velocity w, we can in principle boost the rocket up to the velocity of light in finite time in the observer's system. As a result, there arises for any observer an object similar to a photon, possessing energy, moving with the velocity of light, and having zero mass.

It would seem that the mechanism described of mass boost and generation of corresponding mass-less objects, moving with the velocity of light, can be realized in nature by powerful explosions and different cataclysms, connected with the evolution of starts and with different kinds of jet flows observed in the cosmos.

It is well-known that, in the general theory of relativity, a material particle in free flight moves along a geodesic, this property being retained in the case when the point radiates energy isotropically as a result of the decrease of its rest mass. For instance, in a Schwartzschild field it can be assumed that a material particle with small mass, released in vacuo without initial space velocity $v_0 = 0$, will drop along the geodesic and in finite proper time t^* can expend its entire mass m_0 by radiation, and reach and cross the spatial sphere of the horizon (the boundary of black holes). But, from the observer's point of view, taken at a fixed point of the geodesic world line, the corresponding value of time (the observer's time t) will increase without limit to infinity, when the particle approaches the horizon, where its velocity becomes equal to the velocity of light c.

Obviously, in the presence of an external force G, directed to the boundary of a black hole and accelerating the particle in its motion to the black hole, the observer's time t of the particle reaching the black hole boundary can be finite. It was shown above that, for a rocket, the role of this force G can be played by the rocket thrust resulting from the ejection backwards of its entire mass m_{00}.

The above calculations are made in the context of the ideal theory of rocket motion as a material particle. In a more detailed analysis of accelerations in the distributed rocket mass it is necessary to take account of possible balances of work transformation from the distributed thrust forces, connected with the circuits of the rocket itself, in which different kinds of physico-chemical processes may be realized, accompanied by energy scattering into the ambient medium, and from energy transfer to the ejected masses.

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